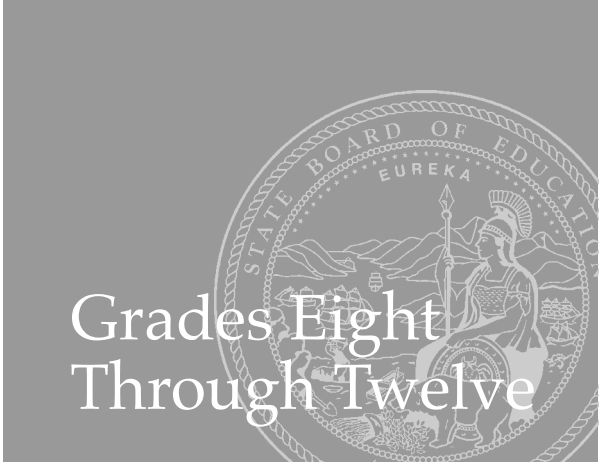

# Grades Eight Through Twelve

## Introduction

The standards for grades eight through twelve are organized differently from those for kindergarten through grade seven. In this section strands are not used for organizational purposes as they are in the elementary grades because the mathematics studied in grades eight through twelve falls naturally under discipline headings: algebra, geometry, and so forth. Many schools teach this material in traditional courses; others teach it in an integrated fashion. To allow local educational agencies and teachers flexibility in teaching the material, the standards for grades eight through twelve do not mandate that a particular discipline be initiated and completed in a single grade. The core content of these subjects must be covered; students are expected to achieve the standards however these subjects are sequenced.

Standards are provided for algebra I, geometry, algebra II, trigonometry, mathematical analysis, linear algebra, probability and statistics, Advanced Placement probability and statistics, and calculus. Many of the more advanced subjects are not taught in every middle school or high school. Moreover, schools and districts have different ways of combining the subject matter in these various disciplines. For example, many schools combine some trigonometry, mathematical analysis, and linear algebra to form a precalculus course. Some districts prefer offering trigonometry content with algebra II.

Table 1, "Mathematics Disciplines, by Grade Level," reflects typical grade-level groupings of these disciplines in both integrated and traditional curricula. The lightly shaded region reflects the minimum requirement for mastery by all students. The dark shaded region depicts content that is typically considered elective but that should also be mastered by students who complete the other disciplines in the lower grade levels and continue the study of mathematics.

**Table 1**

**Mathematics Disciplines, by Grade Level**

| *Discipline* | *Grades* | | | | |
|---|---|---|---|---|---|
| | *Eight* | *Nine* | *Ten* | *Eleven* | *Twelve* |
| Algebra I | | | | | |
| Geometry | | | | | |
| Algebra II | | | | | |
| Probability and Statistics | | | | | |
| Trigonometry | | | | | |
| Linear Algebra | | | | | |
| Mathematical Analysis | | | | | |
| Advanced Placement Probability and Statistics | | | | | |
| Calculus | | | | | |

Many other combinations of these advanced subjects into courses are possible. What is described in this section are standards for the academic content by discipline; this document does not endorse a particular choice of structure for courses or a particular method of teaching the mathematical content.

When students delve deeply into mathematics, they gain not only conceptual understanding of mathematical principles but also knowledge of, and experience with, pure reasoning. One of the most important goals of mathematics is to teach students logical reasoning. The logical reasoning inherent in the study of mathematics allows for applications to a broad range of situations in which answers to practical problems can be found with accuracy.

By grade eight, students' mathematical sensitivity should be sharpened. Students need to start perceiving logical subtleties and appreciate the need for sound mathematical arguments before making conclusions. As students progress in the

study of mathematics, they learn to distinguish between inductive and deductive reasoning; understand the meaning of logical implication; test general assertions; realize that one counterexample is enough to show that a general assertion is false; understand conceptually that although a general assertion is true in a few cases, it is not true in all cases; distinguish between something being proven and a mere plausibility argument; and identify logical errors in chains of reasoning.

Mathematical reasoning and conceptual understanding are not separate from content; they are intrinsic to the mathematical discipline students master at more advanced levels.

# Algebra I

**Symbolic reasoning and calculations with symbols are central in algebra. Through the study of algebra, a student develops an understanding of the symbolic language of mathematics and the sciences. In addition, algebraic skills and concepts are developed and used in a wide variety of problem-solving situations.**

**1.0** Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable:

1.1 Students use properties of numbers to demonstrate whether assertions are true or false.

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**2.0** Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

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**3.0** Students solve equations and inequalities involving absolute values.

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**4.0** Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x\text{-}5) + 4(x\text{-}2) = 12$.

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**5.0** Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

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**6.0** Students graph a linear equation and compute the $x$- and $y$-intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

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**7.0** Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

**8.0** Students understand the concepts of parallel lines and perpendicular lines and how those slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

**9.0** Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

**10.0** Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

**11.0** Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

**12.0** Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

**13.0** Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

**14.0** Students solve a quadratic equation by factoring or completing the square.

**15.0** Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

**16.0** Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

**17.0** Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

**18.0** Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.

**19.0** Students know the quadratic formula and are familiar with its proof by completing the square.

**20.0** Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

**21.0** Students graph quadratic functions and know that their roots are the *x*-intercepts.

**22.0** Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x-axis in zero, one, or two points.

**23.0** Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

**24.0** Students use and know simple aspects of a logical argument:

- 24.1 Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.
- 24.2 Students identify the hypothesis and conclusion in logical deduction.
- 24.3 Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.

**25.0** Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:

25.1 Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.

25.2 Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.

25.3 Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.

## Geometry

**The geometry skills and concepts developed in this discipline are useful to all students. Aside from learning these skills and concepts, students will develop their ability to construct formal, logical arguments and proofs in geometric settings and problems.**

| | |
|---|---|
| **1.0** | Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning. |
| **2.0** | Students write geometric proofs, including proofs by contradiction. |
| **3.0** | Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement. |
| **4.0** | Students prove basic theorems involving congruence and similarity. |
| **5.0** | Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles. |
| **6.0** | Students know and are able to use the triangle inequality theorem. |
| **7.0** | Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles. |
| **8.0** | Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures. |
| **9.0** | Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders. |
| **10.0** | Students compute areas of polygons, including rectangles, scalene triangles, equilateral triangles, rhombi, parallelograms, and trapezoids. |
| **11.0** | Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids. |

**12.0** Students find and use measures of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.

**13.0** Students prove relationships between angles in polygons by using properties of complementary, supplementary, vertical, and exterior angles.

**14.0** Students prove the Pythagorean theorem.

**15.0** Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.

**16.0** Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

**17.0** Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.

**18.0** Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 + (\cos(x))^2 = 1$.

**19.0** Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side.

**20.0** Students know and are able to use angle and side relationships in problems with special right triangles, such as 30°, 60°, and 90° triangles and 45°, 45°, and 90° triangles.

**21.0** Students prove and solve problems regarding relationships among chords, secants, tangents, inscribed angles, and inscribed and circumscribed polygons of circles.

**22.0** Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

# Algebra II

**This discipline complements and expands the mathematical content and concepts of algebra I and geometry. Students who master algebra II will gain experience with algebraic solutions of problems in various content areas, including the solution of systems of quadratic equations, logarithmic and exponential functions, the binomial theorem, and the complex number system.**

**1.0** Students solve equations and inequalities involving absolute value.

**2.0** Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

**3.0** Students are adept at operations on polynomials, including long division.

**4.0** Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

**5.0** Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

**6.0** Students add, subtract, multiply, and divide complex numbers.

**7.0** Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

**8.0** Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

**9.0** Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as $a$, $b$, and $c$ vary in the equation $y = a(x\text{-}b)^2 + c$.

**10.0** Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

**11.0** Students prove simple laws of logarithms.

11.1 Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.

**12.0** Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.

**13.0** Students use the definition of logarithms to translate between logarithms in any base.

**14.0** Students understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values.

**15.0** Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

**16.0** Students demonstrate and explain how the geometry of the graph of a conic section (e.g., asymptotes, foci, eccentricity) depends on the coefficients of the quadratic equation representing it.

**17.0** Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, students can use the method for completing the square to put the equation into standard form and can recognize whether the graph of the equation is a circle, ellipse, parabola, or hyperbola. Students can then graph the equation.

**18.0** Students use fundamental counting principles to compute combinations and permutations.

**19.0** Students use combinations and permutations to compute probabilities.

**20.0** Students know the binomial theorem and use it to expand binomial expressions that are raised to positive integer powers.

**21.0** Students apply the method of mathematical induction to prove general statements about the positive integers.

**22.0** Students find the general term and the sums of arithmetic series and of both finite and infinite geometric series.

**23.0** Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

**24.0** Students solve problems involving functional concepts, such as composition, defining the inverse function and performing arithmetic operations on functions.

**25.0** Students use properties from number systems to justify steps in combining and simplifying functions.

## Trigonometry

**Trigonometry uses the techniques that students have previously learned from the study of algebra and geometry. The trigonometric functions studied are defined geometrically rather than in terms of algebraic equations. Facility with these functions as well as the ability to prove basic identities regarding them is especially important for students intending to study calculus, more advanced mathematics, physics and other sciences, and engineering in college.**

**1.0** Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.

**2.0** Students know the definition of sine and cosine as $y$- and $x$-coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.

**3.0** Students know the identity $\cos^2(x) + \sin^2(x) = 1$:

3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).

3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.

**4.0** Students graph functions of the form $f(t) = A \sin(Bt + C)$ or $f(t) = A \cos(Bt + C)$ and interpret $A$, $B$, and $C$ in terms of amplitude, frequency, period, and phase shift.

**5.0** Students know the definitions of the tangent and cotangent functions and can graph them.

**6.0** Students know the definitions of the secant and cosecant functions and can graph them.

**7.0** Students know that the tangent of the angle that a line makes with the $x$-axis is equal to the slope of the line.

**8.0** Students know the definitions of the inverse trigonometric functions and can graph the functions.

**9.0** Students compute, by hand, the values of the trigonometric functions and the inverse trigonometric functions at various standard points.

**10.0** Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/or simplify other trigonometric identities.

**11.0** Students demonstrate an understanding of half-angle and double-angle formulas for sines and cosines and can use those formulas to prove and/or simplify other trigonometric identities.

**12.0** Students use trigonometry to determine unknown sides or angles in right triangles.

**13.0** Students know the law of sines and the law of cosines and apply those laws to solve problems.

**14.0** Students determine the area of a triangle, given one angle and the two adjacent sides.

**15.0** Students are familiar with polar coordinates. In particular, they can determine polar coordinates of a point given in rectangular coordinates and vice versa.

**16.0** Students represent equations given in rectangular coordinates in terms of polar coordinates.

**17.0** Students are familiar with complex numbers. They can represent a complex number in polar form and know how to multiply complex numbers in their polar form.

**18.0** Students know DeMoivre's theorem and can give *n*th roots of a complex number given in polar form.

**19.0** Students are adept at using trigonometry in a variety of applications and word problems.

## Mathematical Analysis

**This discipline combines many of the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus and strengthens their conceptual understanding of problems and mathematical reasoning in solving problems. These standards take a functional point of view toward those topics. The most significant new concept is that of limits. Mathematical analysis is often combined with a course in trigonometry or perhaps with one in linear algebra to make a year-long precalculus course.**

**1.0** Students are familiar with, and can apply, polar coordinates and vectors in the plane. In particular, they can translate between polar and rectangular coordinates and can interpret polar coordinates and vectors graphically.

**2.0** Students are adept at the arithmetic of complex numbers. They can use the trigonometric form of complex numbers and understand that a function of a complex variable can be viewed as a function of two real variables. They know the proof of DeMoivre's theorem.

**3.0** Students can give proofs of various formulas by using the technique of mathematical induction.

**4.0** Students know the statement of, and can apply, the fundamental theorem of algebra.

**5.0** Students are familiar with conic sections, both analytically and geometrically:

5.1 Students can take a quadratic equation in two variables; put it in standard form by completing the square and using rotations and translations, if necessary; determine what type of conic section the equation represents; and determine its geometric components (foci, asymptotes, and so forth).

5.2 Students can take a geometric description of a conic section—for example, the locus of points whose sum of its distances from (1, 0) and (-1, 0) is 6—and derive a quadratic equation representing it.

**6.0** Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

**7.0** Students demonstrate an understanding of functions and equations defined parametrically and can graph them.

**8.0** Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.